B. Sc Part II (Hons),
Paper III

Rings with zero divisors:

If two non-zero elements a and b of a ring R exist such that $a \cdot b = 0$, then the ring is said to be a ring with zero divisors and a, b are called divisors of zero.

A non-zero element of Ring R is called a zero divisor if there exists a non-zero element b such that $ab = 0$.

Examples of Ring - Which the product of two non-zero elements is zero.

Ex-1: Consider the set of all 2x2 real matrics.

Solution: It has been proved ~~before~~ that the set of all 2x2 real matrices is a ring.

$$\text{Take } A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{and } B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{But } AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Properties of a Ring: If R is a ring with binary operation + and $\cdot$; then for $a, b, c \in R$

(i) $a \cdot 0 = 0 \cdot a = 0$ (ii) $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$

(iii) $(-a) \cdot (-b) = a \cdot b$ (iv) $a \cdot (b - c) = a \cdot b - a \cdot c$ (v) $(b - c) \cdot a = b \cdot a - c \cdot a$

Proof:— (i) We have $a+0=0+a=a$ for every $a\in R$.

in particular, $0+0=0$.

Hence $a\cdot 0=a\cdot(0+0)=a\cdot 0+a\cdot 0$ (by distributive law)

$\Rightarrow a\cdot 0+0=a\cdot 0+a\cdot 0$

$\therefore a\cdot 0=0$

Similarly we can prove $0\cdot a=0$.

(ii) We have $a\cdot\{b+(-b)\}=a\cdot 0$, since $b+(-b)=0$

$=0$ from (i)

Hence by distributive law

$a\cdot b+a\cdot(-b)=0$

Which means that the additive inverse of $a\cdot b$ is $a\cdot(-b)$ that is

$a\cdot(-b)=-(a\cdot b)$

Again $\{a+(-a)\}\cdot b=0,\ b=0$

$\Rightarrow a\cdot b+(-a)\cdot b=0$.

Therefore it follows that $(-a)\cdot b=-(ab)$

Thus (ii) is proved.

(iii) Put $-a=x$, then

$(-a)\cdot(-b)=x\cdot(-b)=-(x\cdot b)$ from (ii)

$=-\{(-a)\cdot b\}$

$=-\{-(a\cdot b)\}=a\cdot b$

(iv) We have $a\cdot(b-c)+a\cdot c=a\cdot\{(b-c)+c\}$ from distributive law

$=a\cdot\{b+(c-c)$ from associative law $=a\cdot(b+0)$

$=a\cdot b+a\cdot 0=a\cdot b+0=a\cdot b$ Therefore $a\cdot(b-c)=a\cdot b-a\cdot c$

(v) It can be proved as in (iv)

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